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A Study of the Fractional Differential Problem of Some Matrix Fractional Functions

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of two matrix fractional functions. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions, matrix fractional functions.**

I. INTRODUCTION

Fractional calculus is a natural extension of classical calculus, which has a history of more than 300 years. In fact, since the birth of differential and integral theory, several mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. However, although much work has been done, the application of fractional derivatives and integrals has only recently begun. In recent years, the development of fractional calculus has stimulated people's new interest in physics, engineering, economics, biology, control theory, and other fields [1-12].

However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [13-17].

In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of two matrix fractional functions. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper.

Definition 2.1 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. On the other hand, for any positive integer p, we define $\left({}_{x_0}D_x^{\alpha}\right)^p[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the p-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([19])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0\ . \tag{3}$$

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Definition 2.3 ([20])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0<\alpha\le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([21])**:** If $0<\alpha\le 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([22])**:** If $0<\alpha\le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{8}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{9}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{10}$$

Definition 2.6: If $0<\alpha\le 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{11}$$

And the matrix α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{12}$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{13}$$

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Theorem 2.7 (matrix fractional Euler's formula)**:** *If* $0 < \alpha \leq 1$, $i = \sqrt{-1}$, *and* A *is a real matrix, then*

$$E_\alpha(iAx^\alpha) = \cos_\alpha(Ax^\alpha) + i\sin_\alpha(Ax^\alpha). \tag{14}$$

Theorem 2.8 (matrix fractional DeMoivre's formula)**:** *If* $0 < \alpha \leq 1$, p *is an integer, and* A *is a real matrix, then*

$$[\cos_\alpha(Ax^\alpha) + i\sin_\alpha(Ax^\alpha)]^{\otimes_\alpha p} = \cos_\alpha(pAx^\alpha) + i\sin_\alpha(pAx^\alpha). \tag{15}$$

Notation2.9: If $z = a + ib$ is a complex number, a, b are real numbers. We denote a the real part of z by $\text{Re}(z)$, and b the imaginary part of z by $\text{Im}(z)$.

III. MAIN RESULTS

In this section, we obtain arbitrary order fractional derivative of two matrix fractional functions. At first, a lemma is needed.

Lemma 3.1: *Suppose that* r, s *are real numbers,* $s > 0$ *and* p *is an integer, then*

$$(r + is)^p = (r^2 + s^2)^{p/2} \exp\left(ip \cdot \operatorname{arccot}\frac{r}{s}\right). \tag{16}$$

Proof $\quad (r + is)^p$

$$= \left[\sqrt{r^2 + s^2}\left(\frac{r}{\sqrt{r^2+s^2}} + i\frac{s}{\sqrt{r^2+s^2}}\right)\right]^p$$

$$= \left(\sqrt{r^2 + s^2}\right)^p (\cos\theta + i\sin\theta)^p \quad \left(\text{where } \theta = \operatorname{arccot}\frac{r}{s}\right)$$

$$= (r^2 + s^2)^{p/2}(\cos p\theta + i\sin p\theta) \quad (\text{by DeMoivre's formula})$$

$$= (r^2 + s^2)^{p/2}\exp(ip\theta) \quad (\text{by Euler's formula})$$

$$= (r^2 + s^2)^{p/2}\exp\left(ip \cdot \operatorname{arccot}\frac{r}{s}\right).$$

q.e.d.

Theorem 3.2: *If* $0 < \alpha \leq 1$, $(-1)^\alpha$ *exists,* r, s *are real numbers,* $s > 0$, p *is any positive integer, and* A *is a real matrix, then the* p*-th order* α*-fractional derivative of the matrix* α*-fractional function*

$$f_\alpha(Ax^\alpha) = E_\alpha(rAx^\alpha) \otimes_\alpha \cos_\alpha(sAx^\alpha) \tag{17}$$

is $\quad \left({}_0D_x^\alpha\right)^p[f_\alpha(Ax^\alpha)]$

$$= (r^2 + s^2)^{p/2} \cdot A^p E_\alpha(rAx^\alpha) \otimes_\alpha \left[\cos\left(p \cdot \operatorname{arccot}\frac{r}{s}\right)\cos_\alpha(sAx^\alpha) - \sin\left(p \cdot \operatorname{arccot}\frac{r}{s}\right)\sin_\alpha(sAx^\alpha)\right]. \tag{18}$$

Proof $\quad \left({}_0D_x^\alpha\right)^p[f_\alpha(Ax^\alpha)]$

$$= \left({}_0D_x^\alpha\right)^p[E_\alpha(rAx^\alpha) \otimes_\alpha \cos_\alpha(sAx^\alpha)]$$

$$= \left({}_0D_x^\alpha\right)^p[\text{Re}\{E_\alpha((r + is)Ax^\alpha)\}]$$

$$= \text{Re}\left\{\left({}_0D_x^\alpha\right)^p[E_\alpha((r + is)Ax^\alpha)]\right\}$$

$$= \text{Re}\{A^p(r + is)^p \cdot E_\alpha((r + is)Ax^\alpha)\}$$

$$= \text{Re}\left\{A^p(r^2 + s^2)^{p/2}\left[\cos\left(p \cdot \operatorname{arccot}\frac{r}{s}\right) + i\sin\left(p \cdot \operatorname{arccot}\frac{r}{s}\right)\right]\left[E_\alpha(rAx^\alpha) \otimes_\alpha [\cos_\alpha(sAx^\alpha) + i\sin_\alpha(sAx^\alpha)]\right]\right\}$$

$$= (r^2 + s^2)^{p/2} \cdot A^p E_\alpha(rAx^\alpha) \otimes_\alpha \left[\cos\left(p \cdot \operatorname{arccot}\frac{r}{s}\right)\cos_\alpha(sAx^\alpha) - \sin\left(p \cdot \operatorname{arccot}\frac{r}{s}\right)\sin_\alpha(sAx^\alpha)\right].$$

q.e.d.

Theorem 3.3: *Suppose that* $0 < \alpha \leq 1$, $(-1)^\alpha$ *exists,* r, s *are real numbers,* $s > 0$, p *is any positive integer, and* A *is a real matrix, then the* p*-th order* α*-fractional derivative of the matrix* α*-fractional function*

$$g_\alpha(Ax^\alpha) = E_\alpha(rAx^\alpha) \otimes_\alpha \sin_\alpha(sAx^\alpha) \tag{19}$$

is $\left({}_{0}D_{x}^{\alpha}\right)^{p}[g_{\alpha}(Ax^{\alpha})]$

$$= (r^2+s^2)^{p/2}\cdot A^p E_\alpha(rAx^\alpha)\otimes_\alpha \left[cos\left(p\cdot arccot\frac{r}{s}\right) sin_\alpha(sAx^\alpha) + sin\left(p\cdot arccot\frac{r}{s}\right) cos_\alpha(sAx^\alpha)\right]. \quad (20)$$

Proof $\left({}_{0}D_{x}^{\alpha}\right)^{p}[g_{\alpha}(Ax^{\alpha})]$

$$= \left({}_{0}D_{x}^{\alpha}\right)^{p}[E_\alpha(rAx^\alpha)\otimes_\alpha sin_\alpha(sAx^\alpha)]$$

$$= \left({}_{0}D_{x}^{\alpha}\right)^{p}[\mathrm{Im}\{E_\alpha((r+is)Ax^\alpha)\}]$$

$$= \mathrm{Im}\left\{\left({}_{0}D_{x}^{\alpha}\right)^{p}[E_\alpha((r+is)Ax^\alpha)]\right\}$$

$$= \mathrm{Im}\{A^p(r+is)^p\cdot E_\alpha((r+is)Ax^\alpha)\}$$

$$= \mathrm{Im}\left\{A^p(r^2+s^2)^{p/2}\left[cos\left(p\cdot arccot\frac{r}{s}\right)+isin\left(p\cdot arccot\frac{r}{s}\right)\right]\left[E_\alpha(rAx^\alpha)\otimes_\alpha[cos_\alpha(sAx^\alpha)+isin_\alpha(sAx^\alpha)]\right]\right\}$$

$$= (r^2+s^2)^{p/2}\cdot A^p E_\alpha(rAx^\alpha)\otimes_\alpha \left[cos\left(p\cdot arccot\frac{r}{s}\right) sin_\alpha(sAx^\alpha) + sin\left(p\cdot arccot\frac{r}{s}\right) cos_\alpha(sAx^\alpha)\right]. \quad \text{q.e.d.}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions, we obtain arbitrary order fractional derivative of two matrix fractional functions. In fact, our results are generalizations of classical calculus results. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in applied mathematics and fractional differential equations.

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